Local estimation of the error in the von Mises' stress and L 2-norm of the stress for linear elasticity problems

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Abstract

Purpose – This paper aims to focus on the local quality of outputs of interest computed by a finite element analysis in linear elasticity.

Design/methodology/approach – In particular outputs of interest are studied which do not depend linearly on the solution of the problem considered such as the $L2$-norm of the stress and the von Mises' stress. The method is based on the concept of error in the constitutive relation.

Findings – The method is illustrated through 2D test examples and shows that the proposed error estimator leads in practice to upper bounds of the output of interest being studied.

Practical implications – This tool is directly usable in the design stage. It can be used to develop efficient adaptive techniques.

Originality/value – The interest of this paper is to provide an estimation of the local quality of $L2$-norm of the stress and the Von Mises' stress as well as practical upper bounds for these quantities.

Keywords Error analysis, Stress (materials), Linear motion, Elasticity

Paper type Research paper

Introduction

The transformation of a continuous mechanical model into a discretized finite element model leads to a partial loss of the information contained in the continuous model and, thus, to the introduction of discretization errors. Methods have been developed over many years in order to evaluate the overall quality of finite element analysis (Babuška and Rheinboldt, 1978; Ladevèze and Pelle, 1983; Zienkiewicz and Zhu, 1987). For linear problems, all these methods provide a global energy-based estimate of the discretization error. Most of the time, such global information is insufficient for dimensioning purposes in mechanical design. In many common situations, the dimensioning criteria involve local values (stresses, displacements, intensity factors …). Developments of error estimators for such quantities were initiated in the 1980s (Babuška and Miller, 1982; Kelly and Isles, 1989). Recently, error estimates and bounds for quantities of interest have been proposed in numerous works (Babuška *et al.*, 1995; Babuška *et al.*, 1998; Rannacher and Stuttmeier, 1997, 1998; Peraire and Patera, 1998; Prudhomme and Oden, 1999; Strouboulis *et al.*, 2000; Ohnimus *et al.*, 2001). In these papers, the quantities of interest depend linearly on the solution of the problem considered, and the computation of the error estimator on the quantity of interest involves the approximate resolution of an auxiliary problem. When the studied quantity not depends linearly on the solution of the problem, authors (Beckers and Rannacher, 2001; Oden and Prudhomme, 2002) propose to linearize the functional of interest. For instance, the quality of the J-integral was studied in (Heintz *et al.*, 2002; Heintz and Samuelsson, 2004; Ruter and Stein, 2006). Another approach, based on the local properties of the error on constitutive relation proposed in (Ladevèze and Rougeot, 1997), was developed in (Ladevèze *et al.*, 1999; Florentin *et al.*, 2003; Florentin *et al.*, 2003; Gallimard and Panetier, 2006). This approach enables the error in the energy norm to be estimated directly in an element E of the mesh.

In this paper, we propose a method to evaluate the error in the $L2$-norm of the stresses in a given zone of the mesh. This method, which can be extended to other quantities, such as the Von Mises' stress, is based on the local properties of the error in constitutive relation proposed in Ladevèze and Rougeot (1997) and on the approximate resolution of an auxiliary problem. In practice, it leads to upper bounds of the quantities being studied.

The paper is structured as follows. In Section 2, the linear elastic problem to be solved is described. In Section 3, we briefly review the basics of the estimation of the error in constitutive relation, and in Section 4 we review the main features of the goal-oriented error estimation. Section 5 focuses on the construction of a local error in the $L2$-norm of the stresses and an associated upper bound. Section 6 shows the extension of this local error estimation to the error in the Von Mises' stress. Finally, numerical examples illustrating the good behavior of these upper bounds and error estimators are proposed in Section 7.

The problem to be solved

Let us consider an elastic structure in a domain Ω bounded by $\partial\Omega$. The external actions on the structure are represented by a prescribed displacement $\underline{u}_d$ over a subset $\partial_1\Omega$ of the boundary, a surface force density $\underline{F}_d$ over $\partial_2\Omega = \partial\Omega - \partial_1\Omega$ and a body force density $\underline{f}_d$ in Ω. K denotes the Hooke's operator of the material. Thus, the problem can be formulated as follows: find a displacement field $\underline{u}$ and a stress field σ defined in Ω which verify:

- the kinematic constraints:

$$\underline{u} \in \mathcal{U} \quad \text{and} \quad \underline{u}|_{\partial_1\Omega} = \underline{u}_d \tag{1}$$

- the equilibrium equations:

$$\begin{gathered}\sigma \in \mathcal{S} \text{ and } \forall \underline{u}^* \in \mathcal{U}_0 \\ -\int_\Omega \sigma : \varepsilon(\underline{u}^*)\mathrm{d}\Omega + \int_\Omega \underline{f}_d \cdot \underline{u}^* \mathrm{d}\Omega + \int_{\partial_2\Omega} \underline{F}_d \cdot \underline{u}^* \mathrm{d}\Gamma = 0\end{gathered} \tag{2}$$

- the constitutive relation:

$$\sigma = K\varepsilon(\underline{u}) \tag{3}$$

where $\mathcal{U}$ denotes the space in which the displacement field is being sought, $\mathcal{S}$ the space of the stresses. $\mathcal{U}_0$ the space of the fields in $\mathcal{U}$ which are zero on $\partial_1\Omega$ and $\varepsilon(\underline{u})$ the linearized strain associated with the displacement $[\varepsilon(\underline{u})]_{ij} = 1/2(u_{i,j} + u_{j,i})$.

The Galerkin finite element method provides an approximation $\underline{u}_h$ of $\underline{u}$ defined in a finite element space $\mathcal{U}_h \in \mathcal{U}$. The space $\mathcal{U}_h$, of finite dimension, is associated with a finite element mesh of characteristic size h. Let P_h denote a decomposition of Ω into N elements E_k. This decomposition is assumed to verify $\bar{\Omega} = \cup_{E_k \in P_h} \bar{E}_k$ with $E_i \cap E_j = \emptyset$ for any i different from j. The discretized problem is:

Find a kinematically admissible finite element displacement field $\underline{u}_h$ such that:

$$\forall \underline{u}_h^* \in \mathscr{U}_{h0}$$
$$-\int_\Omega \mathrm{K}\varepsilon(\underline{u}_h) : \varepsilon(\underline{u}_h^*) \mathrm{d}\Omega + \int_\Omega \underline{f}_d \cdot \underline{u}_h^* \mathrm{d}\Omega + \int_{\partial_2 \Omega} \underline{F}_d \cdot \underline{u}_h^* \mathrm{d}\Gamma = 0 \tag{4}$$

where $\mathscr{U}_{h0}$ is the space of the displacement fields in $\mathscr{U}_h$.

The corresponding stress field is calculated using the constitutive relation:

$$\sigma_h = K\varepsilon(\underline{u}_h) \tag{5}$$

The discretization error $\underline{e}_h$ is the difference between the finite element displacement and the actual displacement of the problem to be solved $\underline{e}_h = \underline{u} - \underline{u}_h$. Traditionally, the energy norm is selected as the error measure.

$$e_h = \left[\int_\Omega K\varepsilon(\underline{u} - \underline{u}_h) : \varepsilon(\underline{u} - \underline{u}_h) \mathrm{d}\Omega \right]^{1/2} \tag{6}$$

The discretization error can also be defined as the error in the stresses:

$$e_h = \left[\int_\Omega (\sigma - \sigma_h) : K^{-1}(\sigma - \sigma_h) \mathrm{d}\Omega \right]^{1/2} \tag{7}$$

The contribution of an element E of the finite element mesh to the global error is:

$$e_{hE} = \left[\int_E K\varepsilon(\underline{u} - \underline{u}_h) : \varepsilon(\underline{u} - \underline{u}_h) \mathrm{d}E \right]^{1/2} = \left[\int_E (\sigma - \sigma_h) : K^{-1}(\sigma - \sigma_h) \mathrm{d}E \right]^{1/2} \tag{8}$$

with the relation:

$$e_h^2 = \sum_{E \in \Omega} e_{hE}^2$$

Error in constitutive relation

Definition

The approach based on the error in constitutive relation relies on a partition of the equations of the problem to be solved into two groups. In linear elasticity, the first group includes the kinematic constraints (1) and the equilibrium equations (2), while the second group contains the constitutive relation (3). Let us consider an approximate solution of the problem, denoted $(\hat{\underline{u}}, \hat{\sigma})$, which satisfies the first group of equations:

The fields $(\hat{\underline{u}}, \hat{\sigma})$ are said to be admissible if:

- the field $\hat{\underline{u}}$ verifies equation (1); and
- the field $\hat{\sigma}$ verifies equation (2)

If $(\hat{\underline{u}}, \hat{\sigma})$ verify the constitutive relation (3) in Ω then $(\hat{\underline{u}}, \hat{\sigma}) = (\underline{u}, \sigma)$. However, if $(\hat{\underline{u}}, \hat{\sigma})$ do not verify the constitutive relation, the quality of this admissible solution can be measured through the residual, denoted $\hat{e}(\hat{\underline{u}}, \hat{\sigma})$, with respect to the verification of the constitutive relation:

$$\hat{e}(\hat{\underline{u}}, \hat{\sigma}) = \hat{\sigma} - K\varepsilon(\hat{\underline{u}}) \tag{9}$$

The associated constitutive relation error for all the elements E of the finite element mesh is:

$$\hat{e}(\hat{\underline{u}}, \hat{\sigma})^2 = \sum_{E \in \Omega} \hat{e}_E(\hat{\underline{u}}, \hat{\sigma})^2 \tag{10}$$

with:

$$\hat{e}_E(\hat{\underline{u}}, \hat{\sigma})^2 = \int_E (\hat{\sigma} - K\varepsilon(\hat{\underline{u}})) : K^{-1}(\hat{\sigma} - K\varepsilon(\hat{\underline{u}}))\mathrm{d}E \tag{11}$$

The relative error $\hat{\varepsilon}$ is then defined by:

$$\hat{\varepsilon}(\hat{\underline{u}}, \hat{\sigma})^2 = \frac{\hat{e}(\hat{\underline{u}}, \hat{\sigma})^2}{\int_\Omega \hat{\sigma}^* K^{-1} \hat{\sigma}^* \mathrm{d}\Omega} \tag{12}$$

with:

$$\hat{\sigma}^* = \frac{1}{2}(\hat{\sigma} + K\varepsilon(\hat{\underline{u}}))$$

The pair $(\underline{u}_h, \sigma_h)$ is not an admissible solution. In order to develop an error estimator based on the concept of error in constitutive relation, we construct an admissible pair $(\hat{\underline{u}}_h, \hat{\sigma}_h)$ from the finite element solution and from the data:

- Since, the finite element displacement field verifies the kinematic constraints, one takes:

$$\hat{\underline{u}}_h = \underline{u}_h \text{ in } \Omega \tag{13}$$

- Conversely, the stress field σ_h does not verify the equilibrium equations (2). Techniques to reconstruct equilibrated stress fields from σ_h and the data have been under development for several years and are described in Ladevèze *et al.* (1991), Ladevèze and Rougeot (1997) and Florentin *et al.* (2002).

Relation with the discretization error

The error in constitutive relation can be related to the discretization error through the Prager and Synge (1947) theorem:

$$
\begin{aligned}
\hat{e}_h^2 &= \hat{e}(\underline{U}_h, \hat{\sigma}_h)^2 \\
&= \int_\Omega (\hat{\sigma}_h - K\varepsilon(\underline{u}_h)) : K^{-1}(\hat{\sigma}_h - K\varepsilon(\underline{u}_h))\mathrm{d}\Omega \\
&= \int_\Omega (\hat{\sigma}_h - \sigma_h) : K^{-1}(\hat{\sigma}_h - \sigma_h)\mathrm{d}\Omega \\
&= \int_\Omega (\hat{\sigma}_h - \sigma) : K^{-1}(\hat{\sigma}_h - \sigma)\mathrm{d}\Omega + \int_\Omega (\sigma - \sigma_h) : K^{-1}(\sigma - \sigma_h)\mathrm{d}\Omega
\end{aligned}
\tag{14}
$$

This theorem leads to the following inequalities:

$$
e_h = \left[\int_\Omega (\sigma - \sigma_h) : \boldsymbol{K}^{-1}(\sigma - \sigma_h)\mathrm{d}\Omega\right]^{1/2} \leq \hat{e}_h \tag{15}
$$

and:

$$
\left[\int_\Omega (\hat{\sigma}_h - \sigma) : K^{-1}(\hat{\sigma}_h - \sigma)\mathrm{d}\Omega\right]^{1/2} \leq \hat{e}_h \tag{16}
$$

A priori, the Prager-Synge theorem cannot be used on the local level. However, in practical situations, when the admissible stress field is built using the techniques developed in Ladevèze and Rougeot (1997), one observes that:

$$
e_{hE} = \|\sigma - \hat{\sigma}_h\|_E \leq C\|\hat{\sigma} - \sigma_h\|_E = C\hat{e}_{hE} \tag{17}
$$

where C is numerically close to 1 (Ladevèze *et al.*, 1999; Florentin *et al.*, 2002; Florentin *et al.*, 2003).

Goal-oriented error estimation

Rather than estimating the numerical error using the energy norm, it would be preferable to calculate the error in terms of physically meaningful quantities of interest. In this section, we briefly review the techniques described in the literature for the case where the quantity of interest is a linear functional L of the displacement. Now, the objective of the calculation is to assess the quality of $I_h = L(\underline{u}_h)$ by estimating $I - I_h$, where $I = L(\underline{u})$. We refer the reader to references (Prudhomme and Oden, 1999; Ladevèze *et al.*, 1999; Strouboulis *et al.*, 2000; Ohnimus *et al.*, 2001; Prudhomme *et al.*, 2003) for a detailed description of the approach. Because of linearity, one has: $I - I_h = L(\underline{u}) - L(\underline{u}_h) = L(\underline{u} - \underline{u}_h) = L(\underline{e}_h)$. Therefore, the estimation of $I - I_h$ is equivalent to the estimation of $L(\underline{e}_h)$.

Definition of the auxiliary problem

Let us consider the following auxiliary problem: find $\underline{z} \in \mathscr{U}_0$ and $\sigma^{\mathrm{aux}} = K\varepsilon(\underline{z})$ such that:

$$
\forall \underline{u}^* \in \mathscr{U}_0 \int_\Omega K\varepsilon(\underline{u}^*) : \varepsilon(\underline{z})\mathrm{d}\Omega = L(\underline{u}^*) \tag{18}
$$

Replacing $\underline{u}^*$ by $\underline{e}_h$, one obtains:

$$L(\underline{e}_h) = \int_\Omega K\varepsilon(\underline{e}_h) : \varepsilon(\underline{z})\mathrm{d}\Omega \tag{19}$$

This relation is the starting point of the goal-oriented error estimates developed in Kelly and Isles (1989), Rannacher and Stuttmeier (1997), Peraire and Patera (1998), Ladevèze *et al.* (1999), Prudhomme and Oden (1999), Strouboulis *et al.* (2000), Ohnimus *et al.* (2001) and Prudhomme *et al.* (2003). The approach proposed here differs in two aspects:

(1) the technique developed to approximate $\underline{z}$; and

(2) the technique used to obtain an upper bound of $|L(\underline{e}_h)|$.

Approximate solution of the auxiliary problem

Let us denote $\underline{z}_h$ the finite element approximation of $\underline{z}$ in the finite element space $\mathcal{U}_{h0}$:

$$\forall \underline{u}_h^* \in \mathcal{U}_{h0} \int_\Omega K\varepsilon(\underline{u}_h^*) : \varepsilon(\underline{z}_h)\mathrm{d}\Omega = L(\underline{u}_h^*) \tag{20}$$

Because of the orthogonality property, one has:

$$\int_\Omega K\varepsilon(\underline{e}_h) : \varepsilon(\underline{u}_h^*)\mathrm{d}\Omega = 0 \quad \forall \underline{u}_h^* \in \mathcal{U}_{h0} \tag{21}$$

Replacing $\underline{u}_h^*$ by $\underline{z}_h$:

$$\int_\Omega K\varepsilon(\underline{e}_h) : \varepsilon(\underline{z}_h)\mathrm{d}\Omega = 0 \tag{22}$$

By combining equations (19) and (22), one gets:

$$I - I_h = L(\underline{e}_h) = \int_\Omega K\varepsilon(\underline{e}_h) : \varepsilon(\underline{z} - \underline{z}_h)\mathrm{d}\Omega = \int_\Omega (\sigma^{\mathrm{aux}} - \sigma_h^{\mathrm{aux}}) : \varepsilon(\underline{e}_h)\mathrm{d}\Omega \tag{23}$$

Let us denote $\hat{\sigma}_h^{\mathrm{aux}}$ an equilibrated stress associated with σ_h^{aux}. The following property is verified:

$$\int_\Omega (\hat{\sigma}_h^{\mathrm{aux}} - \sigma^{\mathrm{aux}}) : \varepsilon(\underline{u}^*)\mathrm{d}\Omega = 0 \quad \forall \underline{u}^* \in \mathcal{U}_0 \tag{24}$$

In particular, replacing $\underline{u}^*$ by $\underline{e}_h$:

$$\int_\Omega (\hat{\sigma}_h^{\mathrm{aux}} - \sigma^{\mathrm{aux}}) : \varepsilon(\underline{e}_h)\mathrm{d}\Omega = 0 \tag{25}$$

By combining equations (25) and (23), one gets:

$$I - I_h = L(\underline{e}_h) = \int_\Omega (\hat{\sigma}_h^{\mathrm{aux}} - \sigma_h^{\mathrm{aux}}) : \varepsilon(\underline{e}_h)\mathrm{d}\Omega \tag{26}$$

Upper bound property

Let us denote $\hat{e}_h^{\text{aux}}$ the error in constitutive relation measured on the auxiliary problem:

$$(\hat{e}_h^{\text{aux}})^2 = \sum_{E \in \Omega} (\hat{e}_{hE}^{\text{aux}})^2 \tag{27}$$

with:

$$(\hat{e}_{hE}^{\text{aux}})^2 = \int_E (\hat{\sigma}^{\text{aux}} - \sigma_h^{\text{aux}}) : K^{-1}(\hat{\sigma}^{\text{aux}} - \sigma_h^{\text{aux}})\mathrm{d}E \tag{28}$$

The Cauchy-Schwartz inequality applied to relation (26) leads to the following two upper bounds:

$$|I - I_h| \leq \hat{e}_h^{\text{aux}} e_h$$

or:

$$|I - I_h| \leq \sum_{E \in \Omega} \hat{e}_{hE}^{\text{aux}} e_{hE} \tag{29}$$

Other bounds were developed in Prudhomme and Oden (1999).

Upper bound of the $L2$-norm of the stress in a zone ω

Let us focus on the quality of the $L2$-norm of the stress in a zone ω of the structure. (In the following discussion, we will assume that ω is the union of some elements E of the finite element mesh.)

$$J = \sqrt{\frac{1}{\text{mes}(\omega)} \int_\omega (\sigma : \sigma)\mathrm{d}\omega} \tag{30}$$

The quantity obtained through the finite element calculation is:

$$J_h = \sqrt{\frac{1}{\text{mes}(\omega)} \int_\omega (\sigma_h : \sigma_h)\mathrm{d}\omega} \tag{31}$$

The quantity being studied does not depend linearly on the finite element solution of the initial problem. The objective is to express the quantity $\delta J = J - J_h$ as a function of the discretization error e_h with respect to the initial problem and of the error in constitutive relation $\hat{e}_h^{\text{aux}}$ calculated for the auxiliary problem. First, we choose to express the difference between the squared quantities $J^2 = J_h^2$.

$$J^2 - J_h^2 = \frac{1}{\text{mes}(\omega)} \left(\int_\omega (\sigma : \sigma - \sigma_h : \sigma_h)\mathrm{d}\omega \right) \tag{32}$$

Thus:

$$J^2 - J_h^2 = \frac{1}{\text{mes}(\omega)} \left(\int_\omega (\sigma - \sigma_h) : (\sigma - \sigma_h)\mathrm{d}\omega + 2\int_\omega \sigma_h : (\sigma - \sigma_h)\mathrm{d}\omega \right) \tag{33}$$

The difference is the sum of two quantities:

$$J^2 - J_h^2 = e_{\text{lin}} + e_{\text{qua}} \tag{34}$$

with:

$$e_{\text{lin}} = \frac{1}{\text{mes}(\omega)} \int_\omega 2\sigma_h : (\sigma - \sigma_h) \text{d}\omega = \int_\omega \frac{2}{\text{mes}(\omega)} K\sigma_h : \varepsilon(\underline{e}_h) \text{d}\omega \tag{35}$$

and:

$$e_{\text{qua}} = \frac{1}{\text{mes}(\omega)} \int_\omega (\sigma - \sigma_h) : (\sigma - \sigma_h) \text{d}\omega \tag{36}$$

$e_{\text{lin}} = L(\underline{e}_h) = I - I_h$ depends linearly on the solution of the initial problem and, thus, e_{lin} can be estimated using the techniques described in Section (5). The quantities I and I_h are defined by:

$$I = \frac{2}{\text{mes}(\omega)} \int_\omega K\sigma_h : \varepsilon(\underline{u}) \text{d}\omega \quad \text{and} \quad I_h = \frac{1}{\text{mes}(\omega)} \int_\omega K\sigma_h : \varepsilon(\underline{u}_h) \text{d}\omega \tag{37}$$

The loading of the associated auxiliary problem (18) is:

$$L(\underline{u}^*) = \int_\Omega \Sigma : \varepsilon(\underline{u}^*) \text{d}\Omega \ \text{ with } \begin{cases} \Sigma = \frac{2}{\text{mes}(\omega)} K\sigma_h & \text{in } \omega \\ \Sigma = 0 & \text{in } \Omega - \omega \end{cases} \tag{38}$$

The auxiliary problem is defined as: find $\underline{z} \in \mathcal{U}_0$ such that:

$$\int_\Omega K\varepsilon(\underline{z}) : \varepsilon(\underline{u}^*) \text{d}\Omega = \int_\Omega \Sigma : \varepsilon(\underline{u}^*) \text{d}\Omega \ \ \forall \underline{u}^* \in \mathcal{U}_0 \tag{39}$$

with:

$$\sigma^{\text{aux}} = K\varepsilon(\underline{z}) \text{ in } \Omega \tag{40}$$

Let $(\underline{z}_h, \sigma_h^{\text{aux}})$ be the finite element solution of the auxiliary problem associated with the finite element space $\mathcal{U}_{h0}$, and let $\hat{\sigma}_h^{\text{aux}}$ be an equilibrated stress field built from σ_h^{aux} and the data (Ladevèze *et al.*, 1991). Furthermore, let $\hat{e}_h^{\text{aux}}$ be the error in the constitutive relation associated with this auxiliary problem. From Property (26), e_{lin} is a function of $\underline{e}_h$:

$$e_{\text{lin}} = L(\underline{e}_h) = \int_\Omega (\hat{\sigma}_h^{\text{aux}} - \sigma_h^{\text{aux}}) : \varepsilon(\underline{e}_h) \text{d}\Omega \tag{41}$$

Using the Cauchy-Schwartz inequality, e_{lin} is bounded by:

$$|e_{\text{lin}}| \leq \sum_{E \in \Omega} \hat{e}_{hE}^{\text{aux}} e_{hE} \tag{42}$$

The second part of the error e_{qua} is an $L2$-norm which can be bounded by the energy norm:

$$
\begin{aligned}
e_{\text{qua}} &= \frac{1}{\text{mes}(\omega)} \int_{\omega} ((\sigma - \sigma_h) : (\sigma - \sigma_h) \text{d}\omega \\
&\leq \frac{1}{\text{mes}(\omega)} \int_{\omega} ((\sigma - \sigma_h) : K^{-1} (\sigma - \sigma_h) \text{d}\omega \quad = \frac{k}{\text{mes}(\omega)} \sum_{E \in \omega} e_{hE}^2
\end{aligned} \tag{43}
$$

Where

$$
k^{-1} = \min_{\sigma \neq 0} \frac{\sigma : K^{-1} \sigma}{\sigma : \sigma}
$$

The upper bound obtained for $J^2 - J_h^2$ is a function of the local contributions e_{hE} of each element E of the mesh to the global discretization error e_h.

$$
J^2 - J_h^2 \leq \eta_{\text{upp},L2}^2 \tag{44}
$$

where:

$$
\eta_{\text{upp},L2}^2 = \sum_{E \in \Omega} \hat{e}_{hE}^{\text{aux}} e_{hE} + \frac{k}{\text{mes}(\omega)} \sum_{E \in \omega} e_{hE}^2 \tag{45}
$$

$$
J \leq \sqrt{J_h^2 + \eta_{\text{upp},L2}^2} \tag{46}
$$

Error estimator and evaluation of an upper bound

The quantities to be estimated are:

$$
\eta_{\text{upp},L2}^2 = \sum_{E \in \Omega} \hat{e}_{hE}^{\text{aux}} e_{hE} + \frac{k}{\text{mes}(\omega)} \sum_{E \in \Omega} e_{hE}^2 \tag{47}
$$

In this expression the only unknowns are the local contributions e_{hE} from each element E.

Property (16) can be used to build a simple, but crude, upper bound:

$$
e_{\text{upp},L2}^2 \leq \hat{e}_h^{\text{aux}} \hat{e}_h + \frac{k}{\text{mes}(\omega)} \hat{e}_h^2
$$

Of course, this upper bound is of little practical interest.

To obtain a better estimate of $\eta_{\text{upp},L2}$, on can use the heuristic property, from equation (17):

$$
\eta_{\text{upp},L2}^2 \leq C \sum_{E \in \Omega} \hat{e}_{hE}^{\text{aux}} \hat{e}_{hE} + C^2 \frac{k}{\text{mes}(\omega)} \sum_{E \in \omega} \hat{e}_{hE}^2
$$

Numerical tests show that C is close to 1 (Ladevèze *et al.*, 1999; Florentin *et al.*, 2002; Florentin *et al.*, 2003). Thus, $\eta_{\text{upp},L2}$ can be estimated by the quantity $\hat{e}_{\text{upp},L2}$:

$$\hat{e}^2_{\mathrm{upp},L2} = \sum_{E\in\Omega} \hat{e}^{\mathrm{aux}}_{hE}\hat{e}_{hE} + \frac{k}{\mathrm{mes}(\omega)} \sum_{E\in\omega} \hat{e}^2_{hE} \tag{48}$$

An estimate of the upper bound of the $L2$-norm of the stress is:

$$\hat{J}_{\mathrm{upp}} = \sqrt{J_h^2 + \hat{e}^2_{\mathrm{upp},L2}} \tag{49}$$

In order to define an error estimator expressed in the quantity of interest, let us observe that in practical situations $\delta J_h = J - J_h$ is small compared to J and J_h, and that we have $J^2 - J_h^2 \approx 2\delta J_h J_h$. The error in the quantity of interest is:

$$\hat{\varepsilon} = \frac{\hat{e}^2_{\mathrm{upp},L2}}{2J_h} = \frac{\sum_{E\in\Omega} \hat{e}^{\mathrm{aux}}_{hE}\hat{e}_{hE} + \frac{k}{\mathrm{mes}(\omega)} \sum_{E\in\omega} \hat{e}^2_{hE}}{2J_h} \tag{50}$$

This expression, in which the error in the quantity of interest is a sum of element contributions, can be used to develop adaptivity procedures through the techniques proposed in (Ladevèze *et al.*, 1991). It is easy to calculate a relative local error by introducing the relative quantity: $\hat{\varepsilon}_{\mathrm{rel}} = \hat{\varepsilon}/J_h$.

Extension to the estimation of the local error in the Von Mises' stress

The Von Mises' stress at a point M of the structure is defined by:

$$\sigma^{\mathrm{VM}} = \sqrt{\frac{3}{2}\,\sigma^D : \sigma^D} \tag{51}$$

where σ^D is the deviatoric part of the stress.

The approximate value of the Von Mises' stress at any point M of the structure is:

$$\sigma_h^{\mathrm{VM}} = \sqrt{\frac{3}{2}\sigma_h^D : \sigma_h^D} \tag{52}$$

The objective is to estimate the quality of the calculated Von Mises' stress in a part ω of the structure. Several quantities of interest can be defined:

$$\begin{aligned} \|\sigma^{\mathrm{VM}}\|_\infty &= \sup_{M\in\omega}|\sigma^{\mathrm{VM}}(M)| \\ \|\sigma^{\mathrm{VM}}\|_{L2} &= \sqrt{\textstyle\int_{M\in\omega}(\sigma^{\mathrm{VM}})^2(M)\mathrm{d}\omega} \\ \|\sigma^{\mathrm{VM}}\|_{L1} &= \textstyle\int_{M\in\omega}|\sigma^{\mathrm{VM}}(M)|\mathrm{d}\omega \end{aligned} \tag{53}$$

The classical relations involving the norms are:

$$\frac{1}{\mathrm{mes}(\omega)}\|\sigma^{\mathrm{VM}}\|_{L1} \le \frac{1}{\sqrt{\mathrm{mes}(\omega)}}\|\sigma^{\mathrm{VM}}\|_{L2} \le \|\sigma^{\mathrm{VM}}\|_\infty \tag{54}$$

If $\sigma^{VM}(M)$ is constant throughout ω, the inequalities are replaced by equalities. Here, we consider the quantity of interest based on the $L2$-norm of the Von Mises' stress, which is:

$$J_{\mathrm{VM}} = \sqrt{\frac{1}{\mathrm{mes}(\omega)} \int_\omega \frac{3}{2} (\sigma^D : \sigma^D) \mathrm{d}\omega} \tag{55}$$

The approximate quantity calculated from the finite element solution is:

$$J_{\mathrm{VM},h} = \sqrt{\frac{1}{\mathrm{mes}(\omega)} \int_\omega \frac{3}{2} (\sigma_h^D : \sigma_h^D) \mathrm{d}\omega} \tag{56}$$

We can follow the same approach as in the previous section by replacing σ by σ^D in the equations. The upper bound used for e_{qua} is the same because the $L2$-norm of the stresses is an upper bound of the $L2$-norm of the Von Mises' stresses. The upper bound for $|e_{\mathrm{lin}}|$ is obtained with a modified auxiliary problem. The expressions of e_{lin} and e_{qua} are:

$$e_{\mathrm{lin}} = \frac{3}{\mathrm{mes}(\omega)} \int_\omega \sigma_h^D : (\sigma - \sigma_h) \mathrm{d}\omega = \int_\omega \frac{3}{\mathrm{mes}(\omega)} K \sigma_h^D : \varepsilon(\underline{e}_h) \mathrm{d}\omega \tag{57}$$

and:

$$e_{\mathrm{qua}} = \frac{1}{\mathrm{mes}(\omega)} \int_\omega \frac{3}{2} (\sigma^D - \sigma_h^D) : (\sigma^D - \sigma_h^D) \mathrm{d}\omega \tag{58}$$

The loading of the auxiliary problem is:

$$L(\underline{u}^*) = \int_\Omega \Sigma : \varepsilon(\underline{u}^*) \mathrm{d}\Omega \ \text{ with } \begin{cases} \Sigma = \frac{3}{\mathrm{mes}(\omega)} & \text{in } \omega \\ \Sigma = 0 & \text{in } \Omega - \omega \end{cases} \tag{59}$$

Let $\eta_{\mathrm{upp,VM}}$ be:

$$\eta_{\mathrm{upp,VM}}^2 = \sum_{E \in \Omega} \hat{e}_{hE}^{aux} e_{hE} + \frac{3k}{2\mathrm{mes}(\omega)} \sum_{E \in \omega} e_{hE}^2 \tag{60}$$

where $\hat{e}_{hE}^{\mathrm{aux}}$ are the local contributions to the error in constitutive relation calculated for the auxiliary problem (20) with the loading defined by equation (59).

The average Von Mises' stress in ω, J_{VM}, is bounded by $\sqrt{J_{\mathrm{VM},h}^2 + \eta_{\mathrm{upp,VM}}^2}$, and $\eta_{\mathrm{upp,VM}}$ is estimated by following the approach developed in Section 5.1 by replacing e_{hE} by $\hat{e}_{hE}$. Then, the error estimator is calculated through equation (50).

Examples

In the numerical example of a plane frame under plane stress (Figure 1), the average norm of the stress and the average Von Mises' stress were calculated in a part ω of the structure. The Young's modulus was $E = 2 \times 10^5$ Mpa and the Poisson's ratio was $\nu = 0.3$ (the coefficient k is equal to 0.769 10^5). The loading and the boundary conditions are described in Figure 1. The initial mesh was made of six-node triangular elements (Figure 2). Two refined meshes were deduced from the initial mesh through

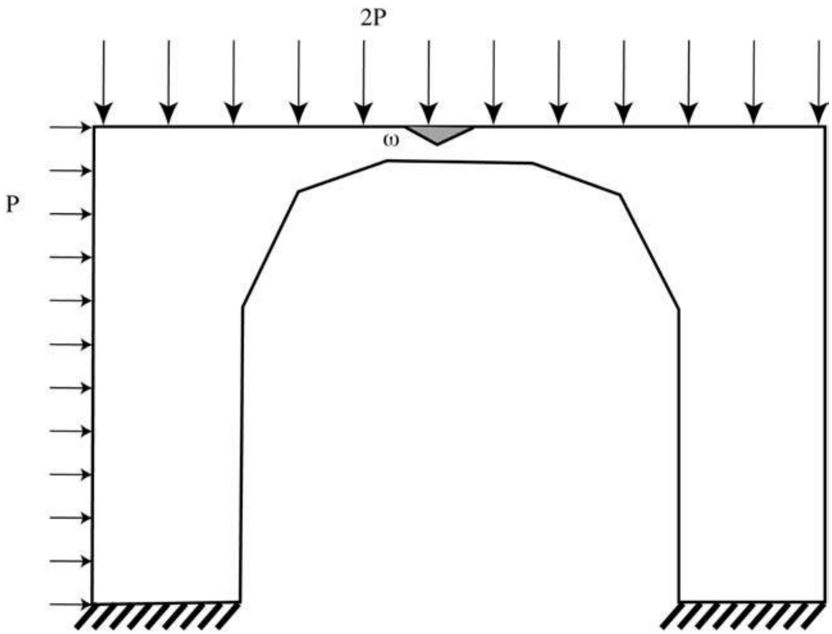

Figure 1.
The problem to be solved

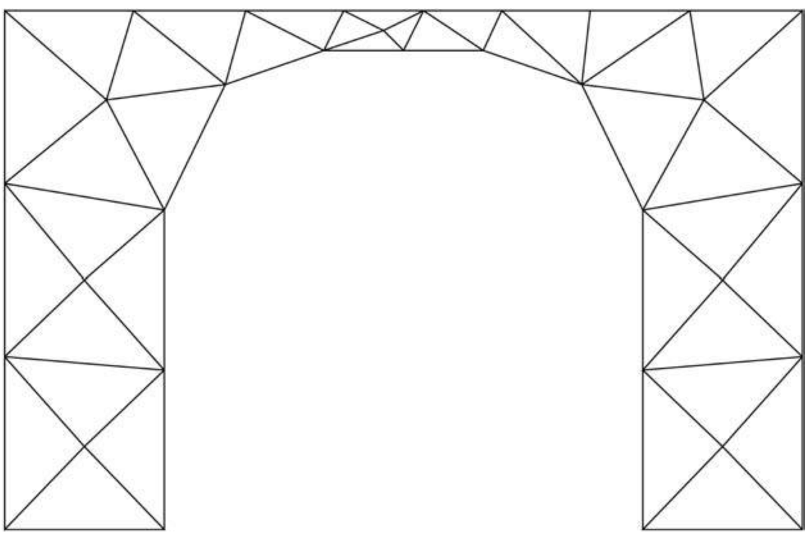

Figure 2.
The initial mesh

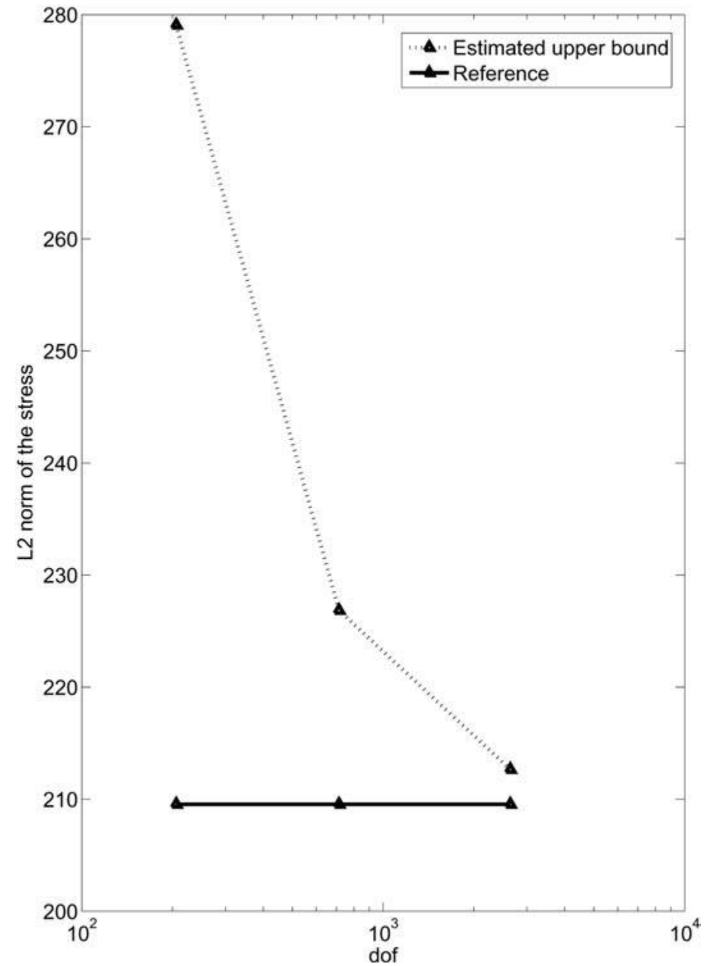

Figure 3.
Upper bound estimate of the *L*2-norm of the stress

two successive uniform refinements. Upper bounds were estimated for the *L*2-norm of the stress and for the Von Mises' stress on these 3 meshes. The reference stress was calculated with the fifth refinement of the initial mesh. The results are shown in Figures 3 and 4. Figure (5) shows the displacements for the auxiliary problem defined in equation (39). Figures 6 and 7 show the ratio between the reference stress and the estimated upper bound. On the second mesh, the stress was overestimated by less than 10 percent. The evolutions of the estimated errors against the reference error (Figures 8 and 9) show that the local estimators overestimate the reference errors by an order of magnitude, which matches the results obtained by Stein *et al.* in (Ohnimus *et al.*, 2001). The evolution of the local relative errors against the global energy error estimator for the initial problem (Figure 10) shows that the local error estimators converge better than the energy error estimator.

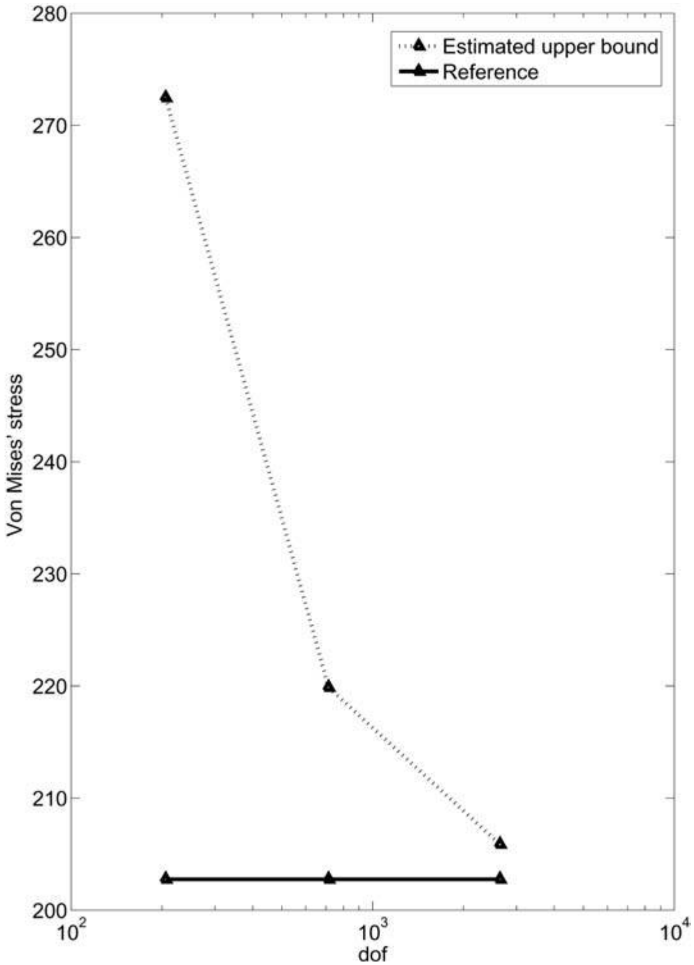

Figure 4.
Upper bound estimate of the Von Mises' stress

As the second example in plane stress, let us consider the average Von Mises' stress in the part ω of the structure described in Figure (11) for various prescribed displacements. Time angle α is equal to $\{0° \ 45° \ 90°\}$. The estimated upper bounds were calculated for the 3 angles (Figure 12). The ratio of the upper bound to a reference Von Mises' stress remained relatively constant (around 1.05) for the 3 loading cases (Figure 13). The comparison between the local relative error estimator and the global error estimator is shown in Figure 14.

Conclusion

We presented a simple local error estimator in the *L*2-norm of the stresses in linear elasticity and we used this error estimator to derive a local error estimator

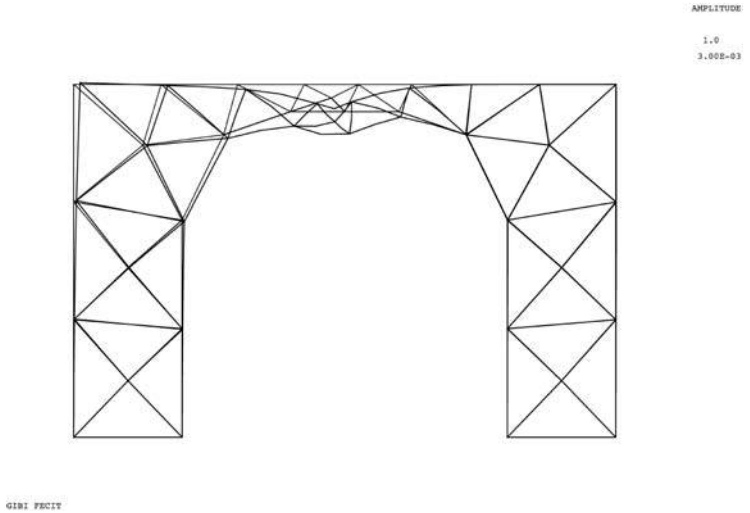

Figure 5.
Calculated displacements for the *L2* auxiliary problem

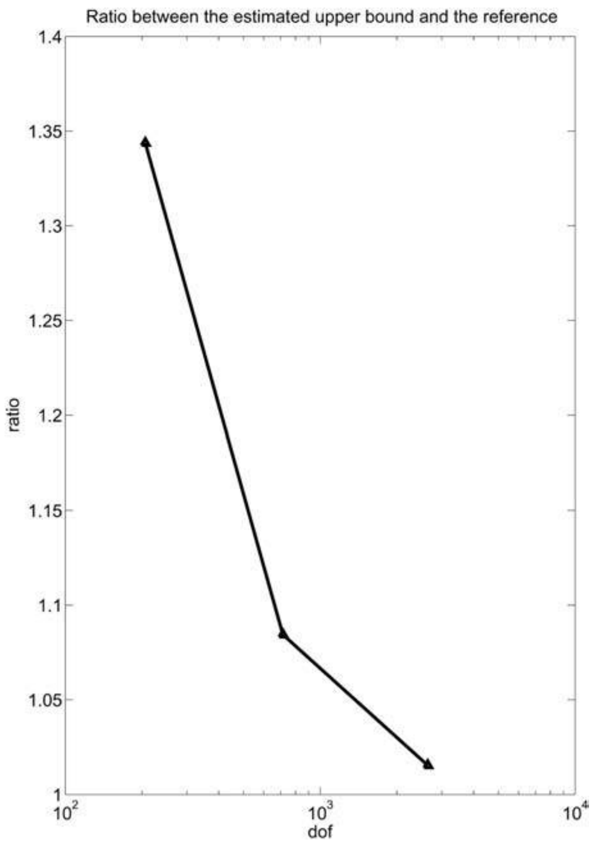

Figure 6.
Ratio of the upper bound to the reference *L2*-norm of the stress

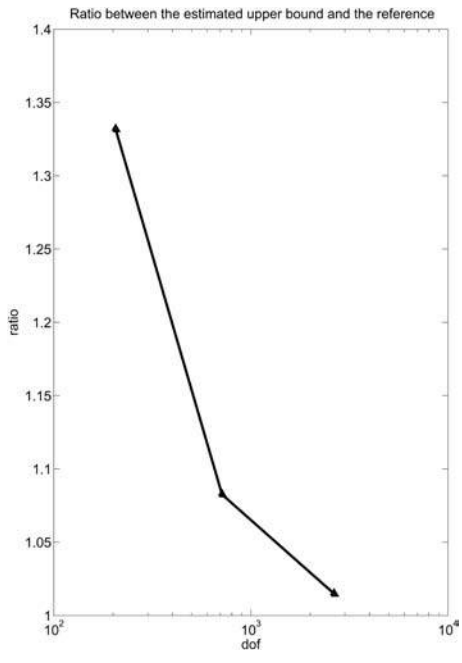

Figure 7.
Ratio of the upper bound to the reference Von Mises' stress

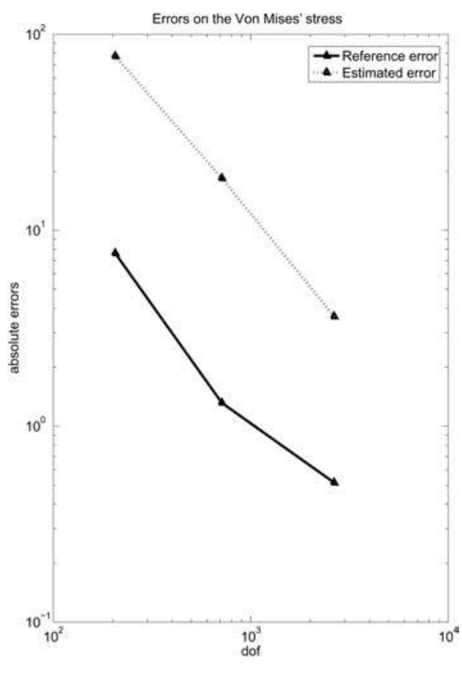

Figure 8.
Von Mises: evolution of the estimated local error

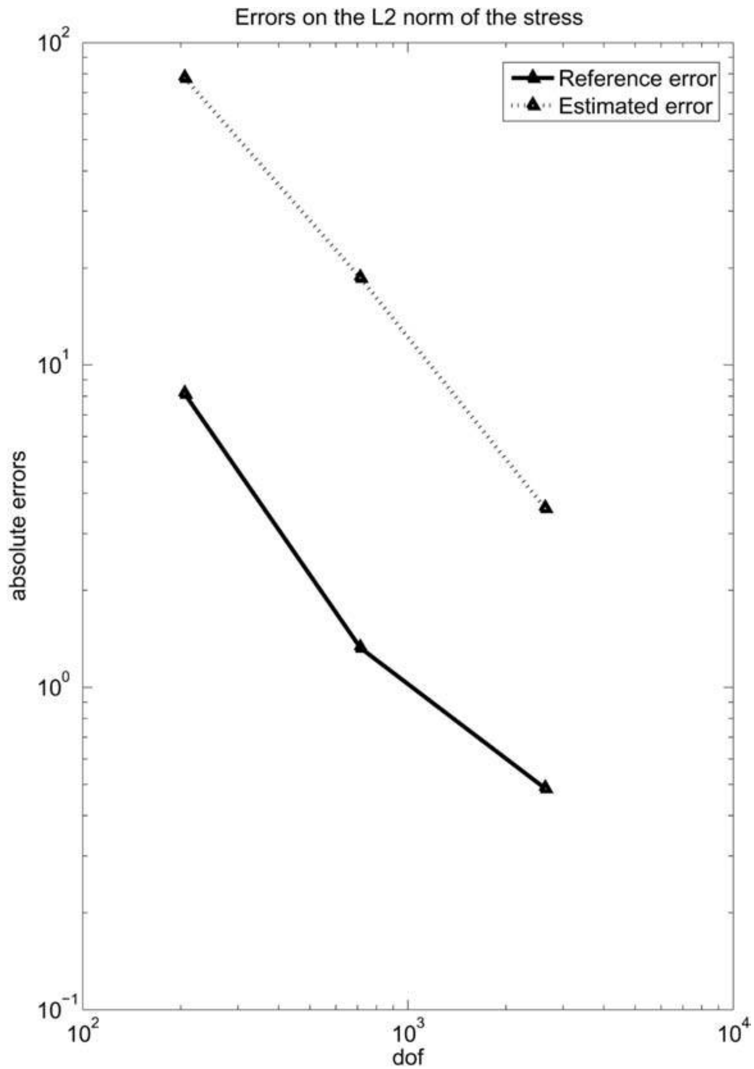

Figure 9.
L2-norm: evolution of the estimated local error

for the Von Mises' stress. The main idea is to combine the classical framework of error estimation for quantities of interest and the practical local properties of the error in constitutive relation. Numerical experiments showed that in practice the proposed error estimators lead to upper bounds of the error and that the behavior of the error estimators is similar to that of the reference errors. Other error estimators can be developed by using the bounds developed in (Prudhomme and Oden, 1999). The error estimators described here can be used in the framework of an adaptive strategy: such a development will be presented by the author in the very near future.

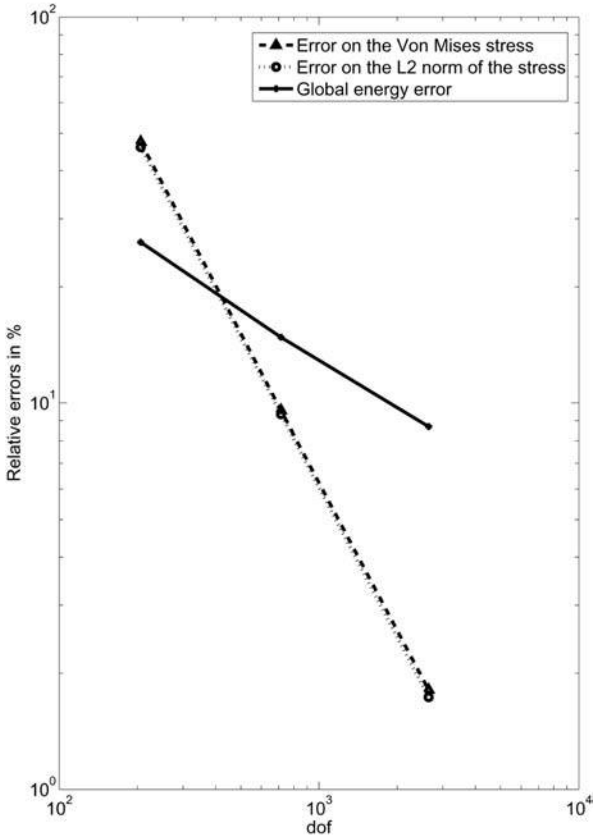

Figure 10.
Evolution of the estimated local errors vs the global energy error estimator for the initial problem

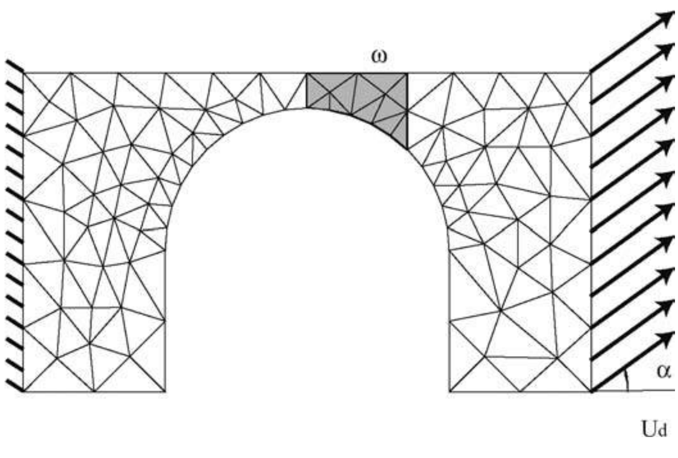

Figure 11.
Problem definition and mesh

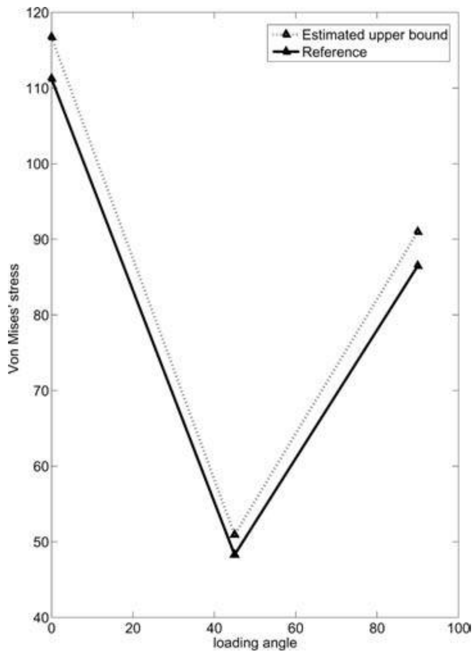

Figure 12.
Upper bound estimate of the Von Mises' stress

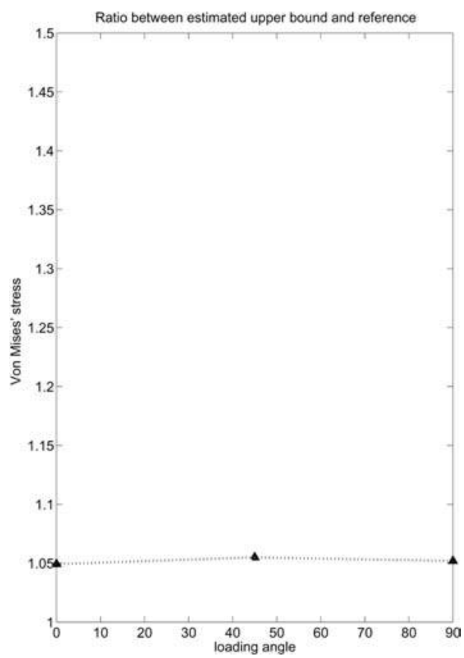

Figure 13.
Ratio of the upper bound to a reference Von Mises' stress

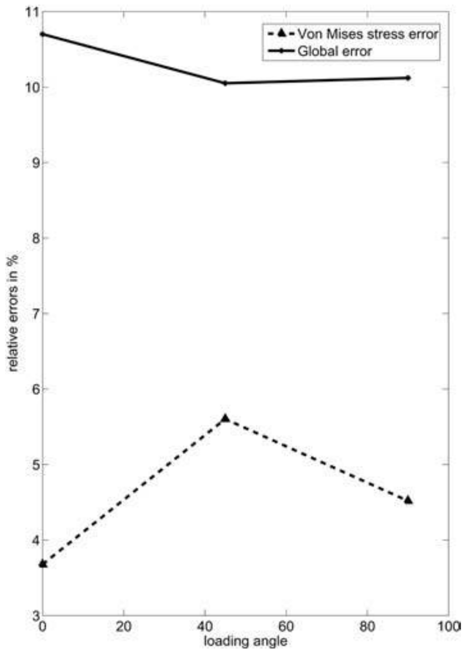

Figure 14.
Estimated local errors vs the global energy error estimator for the initial problem

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Further reading

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